Spacecraft Angular Velocity Stabilization Using A Single-Gimbal Variable Speed Control Moment Gyro

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Feedback controllers for the stabilization of the angular velocity vector of a rigid spacecraft using a single-gimbal Variable Speed Control Moment Gyro (VSCMG) are presented. Linearization of the equations of motion show that complete attitude stabilization is not possible via linear methods. Nonetheless, it is shown that the linearized angular velocity equations are controllable, and a simple LQR control law is used to locally asymptotically stabilize the angular velocity vector. A Lyapunov-based approach is subsequently used to derive a state feedback control law that globally asymptotically stabilizes the nonlinear angular velocity system.

Introduction

Stabilization of the angular velocity equations of a rigid spacecraft with less than three control torques have been addressed in several papers using various techniques. In Ref. 1 it was shown, via Lyapunov methods, that the angular velocity equations can be made locally asymptotically stable about the origin by means of two torques applied along two principal axes. The control law proposed in Ref. 1 was nonlinear. Reference 2 complemented these results by showing asymptotic stability via the construction of a center manifold. A new control law was proposed, and the control law of Ref. 1 was verified. Reference 3 continued this avenue of research by showing that one external torque, aligned with a principal axis, could stabilize the angular velocity vector about the origin. Moreover, it was shown that the controller was robust relative to changes in the parameters defining the control law. In Ref. 4, global asymptotic stability of the angular velocity was proved using a single, linear control law, provided that the spacecraft has no symmetries. It was also shown that a single control torque aligned with a principal axis cannot *asymptotically* stabilize the system. Reference 5 further proved that a body with an axis of symmetry can be globally asymptotically stabilized using one control torque. The resulting control law must necessarily be nonlinear. However, no controller was presented in Ref. 5. Reference 6 verified that the results of Refs. 4 and 5 follow easily as an application of the Jurdjevic-Quinn approach. It also included an explicit nonlinear control law which provided global stability for the axi-symmetric case. Reference 7 showed that the angular velocity of an axi-symmetric rigid body can be globally asymptotically stabilized by means of a linear feedback when two control torques act on the body. Other approaches used to develop globally asymptotically stabilizing controllers for a rigid spacecraft with two torques include the general methodology of nonlinear zero dynamics in Ref. 8, and the energy techniques of Ref. 9. On the same token, the authors of Ref. 10 addressed the angular velocity stabilization of an almost axi-symmetric rigid spacecraft with partial attitude stabilization using two external torques.

In the previously mentioned references, the control torques are assumed to be provided by gas jets. Alternatively, internal torques can be generated by momentum (or reaction) wheels or control moment gyroscopes (CMGs). The spin axis of a momentum wheel is fixed in the body frame, and the spin rate of the flywheel is varied to produce a torque along the spin axis. In the CMG case the wheel speed of the flywheel is kept constant. A gimbal assembly changes the spin axis of the flywheel, thus producing a torque which is orthogonal to both the spin and gimbal axes of the CMG. It is well known that the primary advantage of single-gimbal CMGs over other momentum exchange devices is their torque amplification property, that is, the output torque produced from the rotation of the angular momentum vector is much larger than the one required for gimbal rotation. Several references discuss the use of CMGs for spacecraft attitude control. See, for example, Refs. 11–14.

A complete controllability analysis of the spacecraft equations has been reported in Ref. 15. There, it is

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shown that the system is not controllable with less than three reaction wheels. Krishnan et al[16] provided a control law using two momentum wheels for the restricted case of zero angular momentum. Reference 17 developed a control law to stabilize the spin axis of a rigid spacecraft about a specified inertia axis using two reaction wheels. Finally, the authors in Ref. 18 applied modern nonlinear control techniques for detumbling of a spacecraft with a single momentum wheel aligned along one of the spacecraft principal axes.

The use of Variable Speed Control Moment Gyros (VSCMGs) for spacecraft stabilization has received attention recently.[19,20] A VSCMG can be thought of as a hybrid device comprised of a momentum (or reaction) wheel and a CMG. In particular, the wheel speed of a VSCMG is allowed to vary, thus producing an additional torque over a conventional CMG. This torque is not fixed in the spacecraft body frame, as in the case of a momentum wheel; rather, the direction of the spin axis of the VSCMG is allowed to rotate via a gimbal. An additional torque, perpendicular to the spin and gimbal axes is thus generated, as in the conventional CMG case. This additional degree of freedom can be utilized to avoid the gimbal lock singularity that has plagued traditional CMG clusters.[19] The use of the additional torque of VSCMGs has also been utilized for attitude control (and energy storage) of spacecraft in Refs. 19–21. In both the CMG and VSCMG cases presented in the literature to date, a cluster of actuators has been used to provide a sufficient number of torques to achieve complete attitude stabilization (and possibly energy storage). In this paper, we consider the case of control (stabilization) of a spacecraft via a single VSCMG actuator.

The outline of this paper is as follows. First, we present the complete equations of motion of a spacecraft with one VSCMG in an arbitrary orientation. These equations are composed of the dynamic and kinematic equations. Next, we linearize the equations of motion about an equilibrium point. Linearization shows that the spacecraft attitude is uncontrollable with only one VSCMG. However, the angular velocity equations remain controllable. A simple LQR feedback law is designed to achieve local asymptotic stability at the origin of the linearized angular velocity system. Next, we examine the exact angular velocity equations. We derive a nonlinear control law that ensures global asymptotic stability of the angular velocity of the spacecraft about the origin using only one VSCMG. Several numerical examples are included to demonstrate the viability of the control algorithms proposed.

Equations of Motion

The dynamic equations of motion of a spacecraft with a cluster of VSCMGs have been fully derived in the literature.[19–21] Herein, we will use the equations as derived by Richie et al[21] and Yoon and Tsiotras.[20] In Ref. 20 it is assumed that the center of mass of each VSCMG wheel coincides with that of the gimbal structure; the spacecraft, wheels, and gimbal structure are rigid; the flywheels and gimbals are balanced; and the spacecraft rotational motion is decoupled from its translational motion.

Figure 1 shows a spacecraft with a single VSCMG. The origin of the body frame $\mathcal{B}$, is located at the center of mass of the entire spacecraft. The gimbal frame $\mathcal{G}$, represented by the orthonormal set of unit vectors $\hat{e}_\mathrm{s}$, $\hat{e}_\mathrm{t}$ and $\hat{e}_\mathrm{g}$, is arbitrarily located in the spacecraft platform.

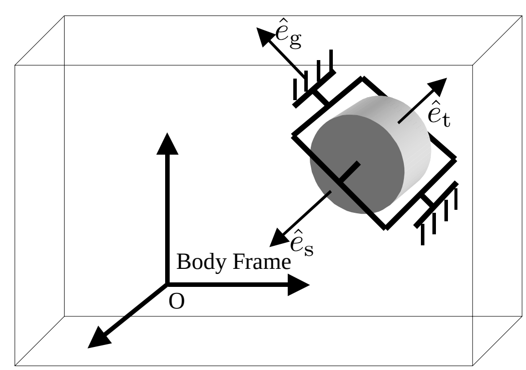

Fig. 1 Spacecraft with a single VSCMG

Dynamics in the body frame

Specializing the dynamical equations of motion presented in Yoon and Tsiotras[20] to a single VSCMG, we get

$$J\dot{\omega}+\dot{J}\omega+A_\mathrm{g}I_\mathrm{cg}\ddot{\gamma}+A_\mathrm{t}I_\mathrm{ws}\Omega\dot{\gamma}+A_\mathrm{s}I_\mathrm{ws}\dot{\Omega}+\omega^\times h=0 \quad (1)$$

where,

$$\begin{aligned} h \quad &:= \quad J\omega + A_\mathrm{g}I_\mathrm{cg}\dot{\gamma} + A_\mathrm{s}I_\mathrm{ws}\Omega && (2)\\ J(\gamma) \quad &:= \quad I^B_B + A_\mathrm{s}(\gamma)I_\mathrm{cs}A_\mathrm{s}^T(\gamma) + A_\mathrm{t}(\gamma)I_\mathrm{ct}A_\mathrm{t}^T(\gamma) \\ &\qquad + A_\mathrm{g}I_\mathrm{cg}A_\mathrm{g}^T && (3)\end{aligned}$$

The argument in $J(\cdot)$ is included to denote explicitly the dependence of the spacecraft inertia matrix on the gimbal axis angle, γ. For notational simplicity, in the sequel we will often drop the argument when it is clear from the context. As a result of (3) it follows that

$$\dot{J} = \dot{\gamma}A_\mathrm{t}(I_\mathrm{cs}-I_\mathrm{ct})A_\mathrm{s}^T + \dot{\gamma}A_\mathrm{s}(I_\mathrm{cs}-I_\mathrm{ct})A_\mathrm{t}^T \quad (4)$$

Notice that $\dot{J}$ is linear in the gimbal rate. In (2) the (column) vector $h \in \mathbb{R}^3$, is the *total* angular momentum of the spacecraft with respect to the inertial frame, expressed in the body frame. Similarly, $\omega \in \mathbb{R}^3$ is the angular velocity vector of the spacecraft with respect to the inertial frame, expressed in the body frame of the spacecraft. The quantities Ω, and $\dot{\Omega}$ are the magnitudes of the angular velocity and angular acceleration of the VSCMG wheel, respectively, about the gimbal spin axis $\hat{e}_{\rm s}$, with respect to the gimbal frame. Also, $\dot{\gamma}$ and $\ddot{\gamma}$ are the magnitudes of the gimbal rate and gimbal acceleration, respectively, about the gimbal axis $\hat{e}_{\rm g}$, with respect to the spacecraft platform. In our analysis we will assume, as usual, gimbal rate commands. This is also the case for standard CMG's in order to take full advantage of the torque amplification property. This implies a velocity steering law for the gimbal. Thus, our control inputs are $\dot{\Omega}$ and $\dot{\gamma}$ and we can write

$$\dot{\gamma} = u_1, \qquad \dot{\Omega} = u_2 \tag{5}$$

Often we use $u := [u_1,\ u_2]^T \in \mathbb{R}^2$ for the combined control vector.

The matrix-valued function $J : [0, 2\pi) \to \mathbb{R}^{3\times 3}$ provides the inertia matrix of the entire spacecraft, i.e. the spacecraft platform, including the wheel and gimbal structure of the VSCMG, given the gimbal angle γ. Notice that the matrix $J(\gamma)$ is positive definite for all values of the gimbal angle $\gamma \in [0, 2\pi)$. The constant matrix I_B^B is the sum of the inertia of the spacecraft platform, and the inertias of the mass centers of the wheel, gimbal and spacecraft platform, about the equivalent mass center of the entire spacecraft. $I_{{\rm c}\star}$ represents the sum of the inertia scalars of the wheel and gimbal structure, i.e. $I_{{\rm c}\star} = I_{{\rm w}\star} + I_{{\rm g}\star}$, where $\star = \{{\rm s}, {\rm t}, {\rm g}\}$.

The column vectors $A_\star \in \mathbb{R}^3$, where $\star = \{{\rm s}, {\rm t}, {\rm g}\}$ are the body frame representations of the gimbal frame unit vectors $\hat{e}_{\rm s}$, $\hat{e}_{\rm t}$, and $\hat{e}_{\rm g}$. Observe that $A_{\rm s}$ and $A_{\rm t}$ are functions of the gimbal angle γ, as follows

$$A_{\rm s}(\gamma) = A_{\rm so} c_\gamma + A_{\rm to} s_\gamma \tag{6a}$$
$$A_{\rm t}(\gamma) = -A_{\rm so} s_\gamma + A_{\rm to} c_\gamma \tag{6b}$$

where $A_{\rm so}$ and $A_{\rm to}$ are the values of $A_{\rm s}$ and $A_{\rm t}$ at some initial time, and $c_\gamma := \cos\gamma$ and $s_\gamma := \sin\gamma$.

Finally, for any vector $v = [v_1,\ v_2\ v_3]^T \in \mathbb{R}^3$, the notation $v^\times \in \mathbb{R}^{3\times 3}$ represents the skew symmetric matrix

$$v^\times = \begin{bmatrix} 0 & -v_3 & v_2 \\ v_3 & 0 & -v_1 \\ -v_2 & v_1 & 0 \end{bmatrix}.$$

For the details of the derivation of equations (1)-(3), as well as the notation used in this paper, the interested reader may refer to Ref. 20.

Kinematics

Without loss of generality, Euler angles will be used to represent the attitude of the spacecraft. For a 3-2-1 Euler angle sequence, the kinematic equations are given by[22]

$$\dot{\phi} = \omega_1 + \omega_2 \sin\phi\tan\theta + \omega_3\cos\phi\tan\theta \tag{7a}$$
$$\dot{\theta} = \omega_2\cos\phi - \omega_3\sin\phi \tag{7b}$$
$$\dot{\psi} = \omega_2\sin\phi\sec\theta + \omega_3\cos\phi\sec\theta \tag{7c}$$

Linear System Analysis

In this section, we linearize the full nonlinear equations of motion, given by (1), (5) and (7), and examine their controllability properties. We also present a linear control law which stabilizes the angular velocity of the linearized system. The only mild assumption made here is that the gimbal acceleration $\ddot{\gamma}$ is negligible.

Linearization

The equilibrium points of Eqs. (1), (5) and (7) are given by $\omega = \phi = \theta = \psi = 0$ and $\gamma = \gamma_o$, $\Omega = \Omega_o$, where γ_o and Ω_o are arbitrary constants. From equations (6) we get

$$A_{\rm s} \approx A_{\rm sf}(\gamma_o) + A_{\rm tf}(\gamma_o)\triangle\gamma \tag{8a}$$
$$A_{\rm t} \approx A_{\rm tf}(\gamma_o) - A_{\rm sf}(\gamma_o)\triangle\gamma, \tag{8b}$$

where

$$A_{\rm sf}(\gamma_o) := A_{\rm so} c_{\gamma_o} + A_{\rm to} s_{\gamma_o}, \tag{9a}$$
$$A_{\rm tf}(\gamma_o) := -A_{\rm so} s_{\gamma_o} + A_{\rm to} c_{\gamma_o}, \tag{9b}$$

and where $\triangle(\cdot)$ represents a small perturbation in the variable from its equilibrium value. Similarly, each term of Eq. (1) results, to first order, in the following terms

$$J\dot{\omega} \approx J_{\rm f}\triangle\dot{\omega} \tag{10a}$$
$$\dot{J}\omega \approx 0 \tag{10b}$$
$$A_{\rm t} I_{\rm ws}\Omega\dot{\gamma} \approx I_{\rm ws}\Omega_o A_{\rm tf}\triangle\dot{\gamma} \tag{10c}$$
$$A_{\rm s} I_{\rm ws}\dot{\Omega} \approx I_{\rm ws} A_{\rm sf}\triangle\dot{\Omega} \tag{10d}$$
$$\omega^\times h \approx -I_{\rm ws}\Omega_o A_{\rm sf}^\times\triangle\omega \tag{10e}$$

where $J_{\rm f}(\gamma_o) := I_B^B + A_{\rm sf} I_{\rm cs} A_{\rm sf}^T + A_{\rm tf} I_{\rm ct} A_{\rm tf}^T + A_{\rm g} I_{\rm cg} A_{\rm g}^T$. The linearization of (1) thus yields

$$\triangle\dot{\omega} = A_1\triangle\omega + B_1\triangle\dot{\gamma} + B_2\triangle\dot{\Omega} \tag{11}$$

where the matrices $A_1 \in \mathbb{R}^{3\times 3}$, $B_1 \in \mathbb{R}^{3\times 1}$, and $B_2 \in \mathbb{R}^{3\times 1}$ are given by

$$A_1(\gamma_o, \Omega_o) := J_{\rm f}^{-1} I_{\rm ws}\Omega_o A_{\rm sf}^\times \tag{12a}$$
$$B_1(\gamma_o, \Omega_o) := -J_{\rm f}^{-1} I_{\rm ws}\Omega_o A_{\rm tf} \tag{12b}$$
$$B_2(\gamma_o, \Omega_o) := -J_{\rm f}^{-1} I_{\rm ws} A_{\rm sf} \tag{12c}$$

Note that these matrices depend on the equilibrium/reference values γ_o and Ω_o. Defining the new state variable as $\triangle x := [\triangle\omega^T, \triangle\phi, \triangle\theta, \triangle\psi]^T \in \mathbb{R}^6$ and the control as $\triangle u := [\triangle\dot{\gamma}, \triangle\dot{\Omega}]^T \in \mathbb{R}^2$, we can express the linearized equations in the familiar form

$$\triangle\dot{x} = A\triangle x + B\triangle u \tag{13}$$

where the matrices $A \in \mathbb{R}^{6\times6}$ and $B \in \mathbb{R}^{6\times2}$ are given by

$$A := \begin{bmatrix} A_1 & 0_{3\times3} \\ \mathbf{1}_{3\times3} & 0_{3\times3} \end{bmatrix}, \quad B := \begin{bmatrix} B_1 & B_2 \\ 0_{3\times1} & 0_{3\times1} \end{bmatrix} \tag{14}$$

where $\mathbf{1}$ is the identity matrix.

Controllability of Linearized System

Here we give two results on the controllability of the linearized complete system in Eqs. (13) and of the linearized angular velocity equations in Eq. (16).

Proposition 1 *The linearized system described by Eqs. (13) and (14) is uncontrollable for any $\gamma_o \in [0, 2\pi)$ and $\Omega_o \in \mathbb{R}$.*

The result follows by showing that the controllability matrix[23]

$$C_o := [B \ AB \ A^2B \ A^3B \ A^4B \ A^5B] \tag{15}$$

has rank five[§] for all $\gamma_o \in [0, 2\pi)$ and $\Omega_o \in \mathbb{R}$. Since the state dimension is six, the pair (A, B) is uncontrollable. □

This result implies that it is not possible to use linear techniques to stabilize the *complete* attitude of the spacecraft using a single VSCMG. It leaves, however, open the possibility that the complete system of equations are controllable in the nonlinear sense.

In the sequel we restrict our attention to the angular velocity subsystem. To this end, define $\tilde{A} := A_1$ and $\tilde{B} := [B_1 \ \ B_2]$, and rewrite Eq. (11) as

$$\triangle\dot{\omega} = \tilde{A}\triangle\omega + \tilde{B}\triangle u \tag{16}$$

Linearizing the kinematic equations, we get

$$\triangle\dot{\phi} = \triangle\omega_1, \qquad \triangle\dot{\theta} = \triangle\omega_2, \qquad \triangle\dot{\psi} = \triangle\omega_3 \tag{17}$$

Proposition 2 *The linearized angular velocity system of Eq. (16) is controllable for all $\gamma_o \in [0, 2\pi)$, and $\Omega_o \neq 0$.*

Proof. From Theorem 3.1 of Ref. 23, $(\tilde{A}, \tilde{B})$ is controllable if the matrix $[\tilde{A} - \lambda\mathbf{1} \,|\, \tilde{B}]$ has full row rank for all $\lambda \in \mathbb{R}$. In particular, this must be true for all the eigenvalues λ_i $(i = 1, 2, 3)$ of the matrix $\tilde{A}$. Calculating the row-reduced echelon form of $\tilde{A}$, one obtains that for $\lambda = \lambda_i$ $(i = 1, 2, 3)$ and $\Omega_o \neq 0$,

$$[\tilde{A} - \lambda\mathbf{1} \,|\, \tilde{B}] = \begin{bmatrix} 1 & 0 & 0 & \diamond & \diamond \\ 0 & 1 & 0 & \diamond & \diamond \\ 0 & 0 & 1 & \diamond & \diamond \end{bmatrix} \tag{18}$$

where $(\diamond)$ are algebraic expressions in terms of the components of $\tilde{A}$ that do not affect the row rank of the matrix. Since the rank of the above matrix is always three, the linearized angular velocity system of Eq. (11) is controllable for all $\gamma_o \in [0, 2\pi)$, and $\Omega_o \neq 0$. ■

LQR Controller for the Angular Velocity Subsystem

Given the controllable system in Eq. (16), we can find a linear control law via LQR methods. For example, we can determine a static full-state feedback law $\triangle u = -K\triangle\omega$ such that the performance cost

$$\mathcal{J} = \int_0^\infty [\triangle\omega^T Q \triangle\omega + \triangle u^T R \triangle u] \, \mathrm{d}t \tag{19}$$

is minimized subject to the dynamics (16). The matrix Q must be positive semi-definite while R must be positive definite. The control gain matrix K is given by $K = R^{-1}\tilde{B}^T P$. The matrix $P = P^T$ is positive semi-definite and satisfies the Algebraic Riccati Equation $\tilde{A}^T P + P\tilde{A} - P\tilde{B}R^{-1}\tilde{B}^T P + Q = 0$. LQR optimal control designs is by now folklore. Details of can be found, for instance, in Ref. 23.

Nonlinear System Analysis

The LQR controller of the previous section ensures asymptotic stability only locally about the equilibrium $\omega = 0$ and for gimbal angles and wheel speeds close to their reference values γ_o and Ω_o, respectively. The last restriction is particularly troublesome, since stabilization of the angular velocity vector should not hinge upon γ and Ω being close to γ_o and Ω_o. As a matter of fact, significant control authority may tend to produce large deviations of the gimbal angle and wheel speed from their reference values; see Eqs. (5). In realistic cases, it is not reasonable to expect that the "states" γ and Ω (whose values are of no particular interest, thus are not penalized in (19)) will remain small. For a more comprehensive analysis of the stabilization problem, it is therefore necessary to work with the exact, nonlinear equations of motion.

In the sequel we improve on the previous results by finding a control law that ensures *global* asymptotic stability for the nonlinear system. We thus also avoid the issue of the restricted (local) validity of the linearized equations due to potentially large deviation of the gimbal angle and wheel speed from their reference values.

As in the linear case, in the sequel we assume that the gimbal acceleration $\ddot{\gamma}$ is negligible, and the control inputs are $\dot{\gamma}$ and $\dot{\Omega}$. The dynamic equations of motion are thus given by

$$J\dot{\omega} + \dot{J}\omega + A_\mathrm{t} I_\mathrm{ws} \Omega\dot{\gamma} + A_\mathrm{s} I_\mathrm{ws} \dot{\Omega} + \omega^\times h = 0 \tag{20}$$

[§]The rank of the matrix C_o was calculated using the Symbolic Toolbox of MATLAB.[24]

where h and J are as in (2) and (3).

To derive a stabilizing control law for (20), we consider the positive definite, continuously differentiable Lyapunov function $V(\omega) := \frac{1}{2}\omega^T J\omega$. The derivative of V along the trajectories of the system is

$$\begin{aligned}\dot{V}(\omega) &= \omega^T J\dot{\omega} + \frac{1}{2}\omega^T \dot{J}\omega \\ &= \omega^T J\dot{\omega} + \omega^T \dot{J}\omega - \frac{1}{2}\omega^T \dot{J}\omega \\ &= \omega^T (J\dot{\omega} + \dot{J}\omega) - \frac{1}{2}\omega^T \dot{J}\omega \\ &= -\omega^T (A_{\mathrm{t}} I_{\mathrm{ws}}\Omega\dot{\gamma} + A_{\mathrm{s}} I_{\mathrm{ws}}\dot{\Omega} + \omega^\times h) - \frac{1}{2}\omega^T \dot{J}\omega\end{aligned}$$

Rewriting $\dot{J} = \Phi\dot{\gamma}$ where $\Phi := A_{\mathrm{t}}(I_{\mathrm{cs}} - I_{\mathrm{ct}})A_{\mathrm{s}}^T + A_{\mathrm{s}}(I_{\mathrm{cs}} - I_{\mathrm{ct}})A_{\mathrm{t}}^T$ and using the fact that $\omega^T\omega^\times h = 0$ yields

$$\begin{aligned}\dot{V}(\omega) &= -\omega^T A_{\mathrm{t}} I_{\mathrm{ws}}\Omega\dot{\gamma} - \frac{1}{2}\omega^T \Phi\omega\dot{\gamma} - \omega^T A_{\mathrm{s}} I_{\mathrm{ws}}\dot{\Omega} \\ &= -\omega_{\mathrm{t}} I_{\mathrm{ws}}\Omega\dot{\gamma} - \omega_{\mathrm{s}}\omega_{\mathrm{t}}(I_{\mathrm{cs}} - I_{\mathrm{ct}})\dot{\gamma} - I_{\mathrm{ws}}\omega_{\mathrm{s}}\dot{\Omega}\end{aligned}$$

where $\omega_{\mathrm{s}} = \omega^T A_{\mathrm{s}}$ and $\omega_{\mathrm{t}} = \omega^T A_{\mathrm{t}}$ are the components of the body angular velocity vector $\vec{\omega}$ along the spin and transverse axes of the gimbal frame, respectively, i.e. $\vec{\omega} = \omega_{\mathrm{s}}\hat{e}_{\mathrm{s}} + \omega_{\mathrm{t}}\hat{e}_{\mathrm{t}} + \omega_{\mathrm{g}}\hat{e}_{\mathrm{g}}$.

Proposition 3 *Consider the following control law*

$$\begin{aligned}\dot{\gamma} &= k_1(\Omega)\,\omega_{\mathrm{t}}\left(\Omega - \omega_{\mathrm{s}}\left(\frac{I_{\mathrm{cs}} - I_{\mathrm{ct}}}{I_{\mathrm{ws}}}\right)\right) \\ \dot{\Omega} &= k_2\omega_{\mathrm{s}} + k_3|\omega_{\mathrm{t}}|k_4(\Omega) + k_1(\Omega)\left(\frac{I_{\mathrm{cs}} - I_{\mathrm{ct}}}{I_{\mathrm{ws}}}\right)^2 \omega_{\mathrm{t}}^2\omega_{\mathrm{s}}\end{aligned}$$

where $k_1 : \mathbb{R} \to \mathbb{R}_+$ is any function such that $k_1(\Omega)\Omega^2$ is bounded for all $\Omega \in \mathbb{R}$, $k_4(\Omega) := \Omega k_1^{\frac{1}{2}}(\Omega)$, $k_2 > 0$ and $2\sqrt{k_2} > k_3 \geq 0$. This control law globally asymptotically stabilizes the system given by Eq. (20) for all $\Omega(0) \neq 0$.

Proof. Substituting this control law in the expression for $\dot{V}(\omega)$ leads to

$$\begin{aligned}\dot{V}(\omega) = &-\omega_{\mathrm{t}}^2\Omega I_{\mathrm{ws}}k_1(\Omega)\left(\Omega - \omega_{\mathrm{s}}\left(\frac{I_{\mathrm{cs}} - I_{\mathrm{ct}}}{I_{\mathrm{ws}}}\right)\right) \\ &-\omega_{\mathrm{s}}\omega_{\mathrm{t}}^2(I_{\mathrm{cs}} - I_{\mathrm{ct}})k_1(\Omega)\left(\Omega - \omega_{\mathrm{s}}\left(\frac{I_{\mathrm{cs}} - I_{\mathrm{ct}}}{I_{\mathrm{ws}}}\right)\right) \\ &-I_{\mathrm{ws}}\omega_{\mathrm{s}}\Big(k_2\omega_{\mathrm{s}} + k_3|\omega_{\mathrm{t}}|k_4(\Omega) \\ &\qquad + k_1(\Omega)\left(\frac{I_{\mathrm{cs}} - I_{\mathrm{ct}}}{I_{\mathrm{ws}}}\right)^2 \omega_{\mathrm{t}}^2\omega_{\mathrm{s}}\Big) \\ = &-\omega_{\mathrm{t}}^2 I_{\mathrm{ws}}\Omega^2 k_1(\Omega) - k_2 I_{\mathrm{ws}}\omega_{\mathrm{s}}^2 - k_3 I_{\mathrm{ws}}\omega_{\mathrm{s}}|\omega_{\mathrm{t}}|k_4(\Omega) \\ = &-I_{\mathrm{ws}}\begin{bmatrix}\omega_{\mathrm{t}} & \omega_{\mathrm{s}}\end{bmatrix} G(\Omega,\omega,\gamma)\begin{bmatrix}\omega_{\mathrm{t}} \\ \omega_{\mathrm{s}}\end{bmatrix}\end{aligned}$$

where the matrix $G(\Omega,\omega,\gamma)$ is given by

$$G(\Omega,\omega,\gamma) := \begin{bmatrix}\Omega^2 k_1(\Omega) & \frac{k_3}{2}\mathrm{sgn}(\omega_{\mathrm{t}})k_4(\Omega) \\ \frac{k_3}{2}\mathrm{sgn}(\omega_{\mathrm{t}})k_4(\Omega) & k_2\end{bmatrix}$$

It can be easily shown that $G(\Omega,\omega,\gamma) \geq 0$ for all $(\Omega,\omega,\gamma) \in \mathbb{R}^2 \times [0, 2\pi)$ and $G(\Omega,\omega,\gamma) > 0$ for $\Omega \neq 0$. It follows that $\dot{V} \leq 0$. The last inequality shows that V, and hence ω is bounded. Therefore, $\dot{\gamma}$ and $\dot{\Omega}$ as well as $\dot{\gamma}\Omega$ are bounded. Moreover, $\dot{\omega}$ is bounded from (20). It follows that ω, γ and Ω are uniformly continuous and thus $\dot{V}$ is uniformly continuous as well. From Barbalat's Lemma[25] it follows that $\dot{V} \to 0$. This implies that $\omega_{\mathrm{s}} \to 0$ and $\Omega\omega_{\mathrm{t}} \to 0$ as $t \to \infty$. Assume now that $\Omega(0) \neq 0$ and that $\omega_{\mathrm{t}} \to \bar{\omega}_{\mathrm{t}} \neq 0$. Since $\omega_{\mathrm{s}} \to 0$ we then have that after a sufficiently long time, $\dot{\Omega} \approx k_3|\bar{\omega}_{\mathrm{t}}|\Omega k_1^{\frac{1}{2}}(\Omega)$ and the equilibrium $\Omega = 0$ is unstable. Hence, $\Omega\omega_{\mathrm{t}} \not\to 0$, a contradiction. It follows that, necessarily, $\omega_{\mathrm{t}} \to 0$ as $t \to \infty$.

Assume now that $\omega_{\mathrm{s}} = \Omega\omega_{\mathrm{t}} \equiv 0$. It follows that $\dot{\gamma} = \dot{\Omega} = 0$ and from Eq. (20)

$$J\dot{\omega} + \omega^\times(J\omega + A_{\mathrm{s}} I_{\mathrm{ws}}\Omega) = 0 \tag{21}$$

which, when expressed in the gimbal frame, becomes

$$\begin{aligned}J_{13}\dot{\omega}_{\mathrm{g}} - J_{23}\omega_{\mathrm{g}}^2 &= 0 &(22a)\\ J_{23}\dot{\omega}_{\mathrm{g}} + J_{13}\omega_{\mathrm{g}}^2 + I_{\mathrm{ws}}\Omega\omega_{\mathrm{g}} &= 0 &(22b)\\ J_{33}\dot{\omega}_{\mathrm{g}} &= 0 &(22c)\end{aligned}$$

From Eq. (22c), we get $\dot{\omega}_{\mathrm{g}} = 0$. From Eq. (22a) or Eq. (22b), we conclude that $\omega_{\mathrm{g}} = 0$. Thus, the largest invariant set in $\{\omega : \dot{V}(\omega) = 0\}$ is the set $\omega = 0$. Asymptotic stability follows from LaSalle's theorem and global asymptotic stability follows from the radial unboundedness of the function V and the fact that the previous analysis holds for all initial conditions $\omega \in \mathbb{R}^3$. ■

Acceleration Steering Law

In the actual spacecraft the gimbal control input is a torque (or gimbal acceleration) command, rather than a gimbal velocity command. The derived velocity command has to be implemented via an internal servo control loop. A simple implementation of this idea is to use, say

$$\ddot{\gamma} = K_{\mathrm{p}}(\dot{\gamma}_{\mathrm{d}} - \dot{\gamma}) \tag{23}$$

where $K_{\mathrm{p}} > 0$ and where $\dot{\gamma}_{\mathrm{d}}$ as in Proposition 3. This (proportional) control law will ensure that the actual gimbal velocity $\dot{\gamma}$ approaches the desired command $\dot{\gamma}_{\mathrm{d}}$, as $t \to \infty$. In practice K_{p} has to be sufficiently large in order for the convergence to take place in a short interval of time.

Numerical Examples

In this section we give some illustrative examples of the control design methods for the angular velocity subsystem using both the linear and the nonlinear analysis of the previous sections. Both examples applied the control laws developed earlier to the *complete* equations of motion in Eqs. (1)-(4) using the acceleration steering law in Eq. (23). This was done in order to compare each control law individually and in relation to each other using a realistic evaluation model. Table 1 summarizes the values the moments of inertia and gimbal used in the simulations. These values roughly correspond to the spacecraft simulator described in Ref. 26. The controller gains and the initial conditions are given in Table 2.

The first example corresponds to the LQR control design method which was developed from the linearized angular velocity system. The results are shown in Fig. 2. The weighting matrices Q and R in (19) were chosen by trial and error to stabilize the system quickly with suitable damping. Their values are shown in Table 2. Figures 3 and 4 show the values of the gimbal angle and wheel speed as well as their rates.

Table 1 Moments of inertia values.

Symbol	Value	Units
I_B^B	$\begin{bmatrix} 15.303 & 3.0 & 4.0 \\ 3.0 & 13.224 & 2.0 \\ 4.0 & 2.0 & 19.903 \end{bmatrix}$	$\mathrm{kg\,m^2}$
I_{ws}	0.0042	$\mathrm{kg\,m^2}$
$I_{\mathrm{wt}}, I_{\mathrm{wg}}$	0.0024	$\mathrm{kg\,m^2}$
I_{gs}	0.0093	$\mathrm{kg\,m^2}$
$I_{\mathrm{gt}}, I_{\mathrm{gg}}$	0.0054	$\mathrm{kg\,m^2}$
A_{so}	$[-1, 0, 0]^T$	–
A_{to}	$[0, 0.8161, 0.5779]^T$	–
A_{go}	$[0, 0.5779, -0.8161]^T$	–

The second example corresponds to the nonlinear control law of Proposition 3. The function $k_1(\Omega)$ was chosen as $k_1(\Omega) = \mu/(1 + \Omega^2)$. The angular velocity histories with the nonlinear control law are shown in Fig. 5. The time history of $\dot\gamma$ and $\dot\Omega$ are shown in Fig. 6. The time histories of the gimbal angles and the wheel speed velocity are shown in Fig. 7.

Notice that the nonlinear controller is more aggressive resulting in larger values for the gimbal angle and wheel speed. Since in a physical system the wheel and the gimbal rate commands saturate, it is imperative to modify the nonlinear control law so as to take into account these saturation effects. This is left for future investigation.

On the other hand, one may choose to use the nonlinear controller only if the initial conditions be-

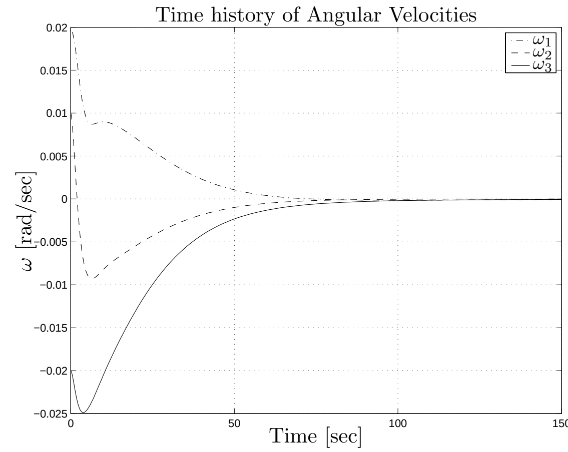

Fig. 2 Numerical simulations with the LQR control law.

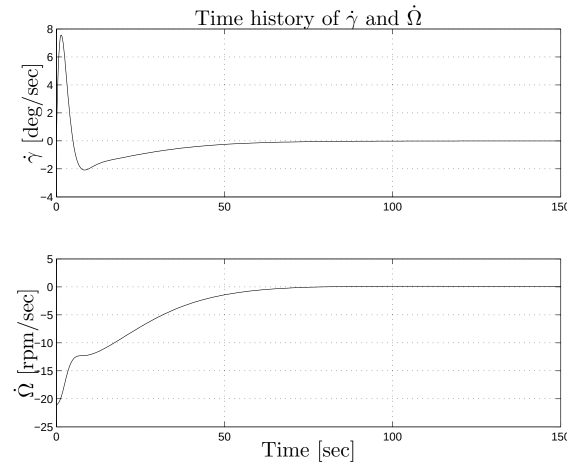

Fig. 3 Time history of $\dot\gamma$ and $\dot\Omega$ with the LQR control law.

Table 2 Controller gains and initial conditions.

Symbol	Linear	Nonlinear	Units
$\omega(0)$	$[0.02, 0.01, -0.02]^T$	$[0.1, 0.1, -0.1]^T$	rad/sec
$\gamma(0)$	$\gamma_o = 20$	20	deg
$\dot\gamma(0)$	0	0	deg
$\Omega(0)$	$\Omega_o = 2 \times 10^3$	2×10^3	rpm
Q	$\mathrm{diag}\{1\mathrm{e}^4, 1\mathrm{e}^4, 1\mathrm{e}^4\}$	–	–
R	$\mathrm{diag}\{1\mathrm{e}^2, 1\}$	–	–
μ	–	800	sec^{-1}
k_2	–	400	sec^{-1}
k_3	–	10	$\mathrm{sec}^{-\frac{1}{2}}$
K_{p}	1	1	sec^{-1}

come too large. After the trajectories reach a small neighborhood of the origin (and within the region of attraction of the linear controller), one can then switch to the LQR controller, whose local performance can be pre-assigned via the optimization criterion (19). In

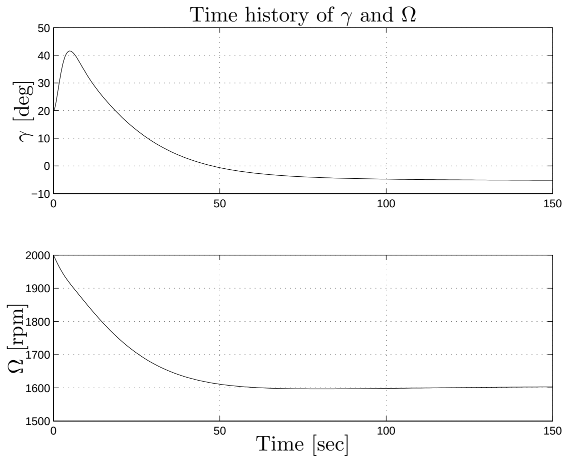

Fig. 4 Time history of γ and Ω with the LQR control law.

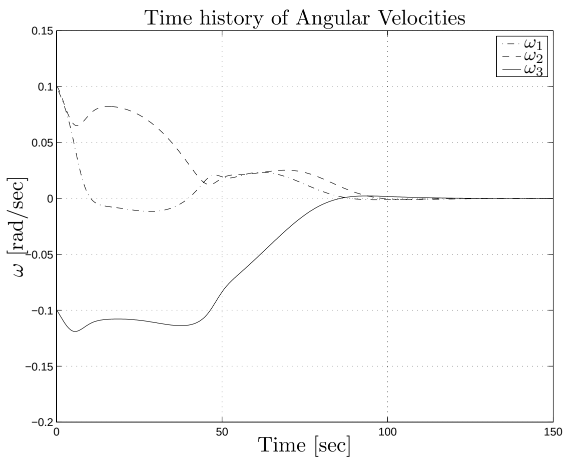

Fig. 5 Numerical simulations with the nonlinear control law.

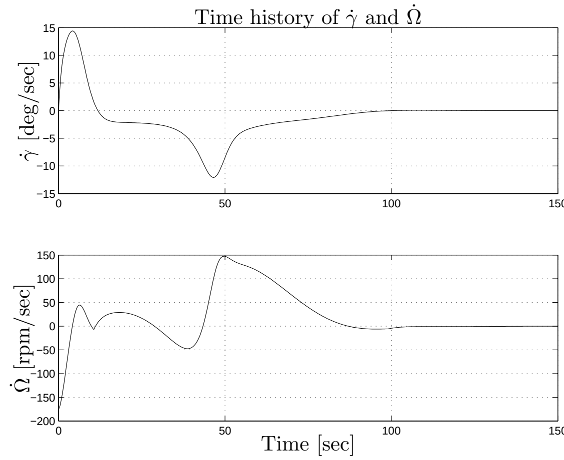

Fig. 6 Time history of $\dot{\gamma}$ and $\dot{\Omega}$ with the nonlinear control law.

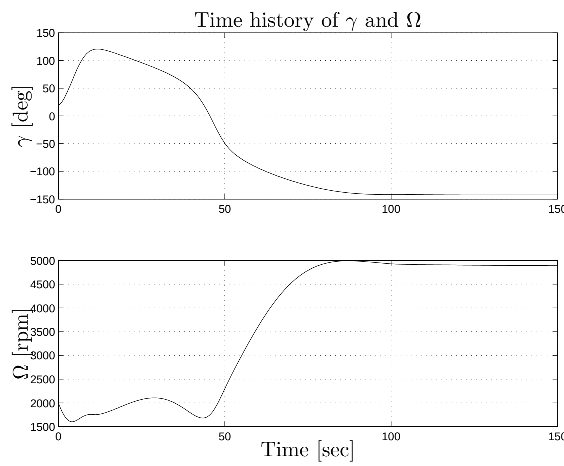

Fig. 7 Time history of γ and Ω with the nonlinear control law.

this sense, the linear controller achieves (local) performance and stability, whereas the nonlinear controller acts as a "safety net" to protect the system from large initial conditions.

For comparison, in Fig. 8 we show the results from the numerical simulations of the LQR with initial conditions $\omega(0) = [0.1\ 0.1\ {-0.1}]^T$. For these (large) initial conditions, the LQR does not stabilize the nonlinear system.

Finally, Fig. 9 shows a series of snapshots of a spacecraft with one VSCMG undergoing a detumbling maneuver using the nonlinear control law of Proposition 3. Note that, as expected, the final orientation of the spacecraft is such that the spin axis of the VSCMG is aligned with the total angular momentum vector, which remains constant in inertial frame at all times.

Conclusions

In this paper, we have addressed the stabilization problem of a rigid spacecraft with a single-gimbal variable-speed control moment gyro (VSCMG). Since no external control torques act on the system, reorientation of the spacecraft is achieved via momentum transfer between the spacecraft platform and the VSCMG. We showed that the complete attitude equations are not linearly controllable. The angular velocity equations are, nonetheless controllable. A simple LQR controller was used to locally asymptotically stabilize the angular velocity equations for an arbitrary gimbal frame orientation. Abandoning the restrictive assumptions made in the linear case we developed a control law for the nonlinear system which ensures global asymptotic stability of the angular velocity equations.

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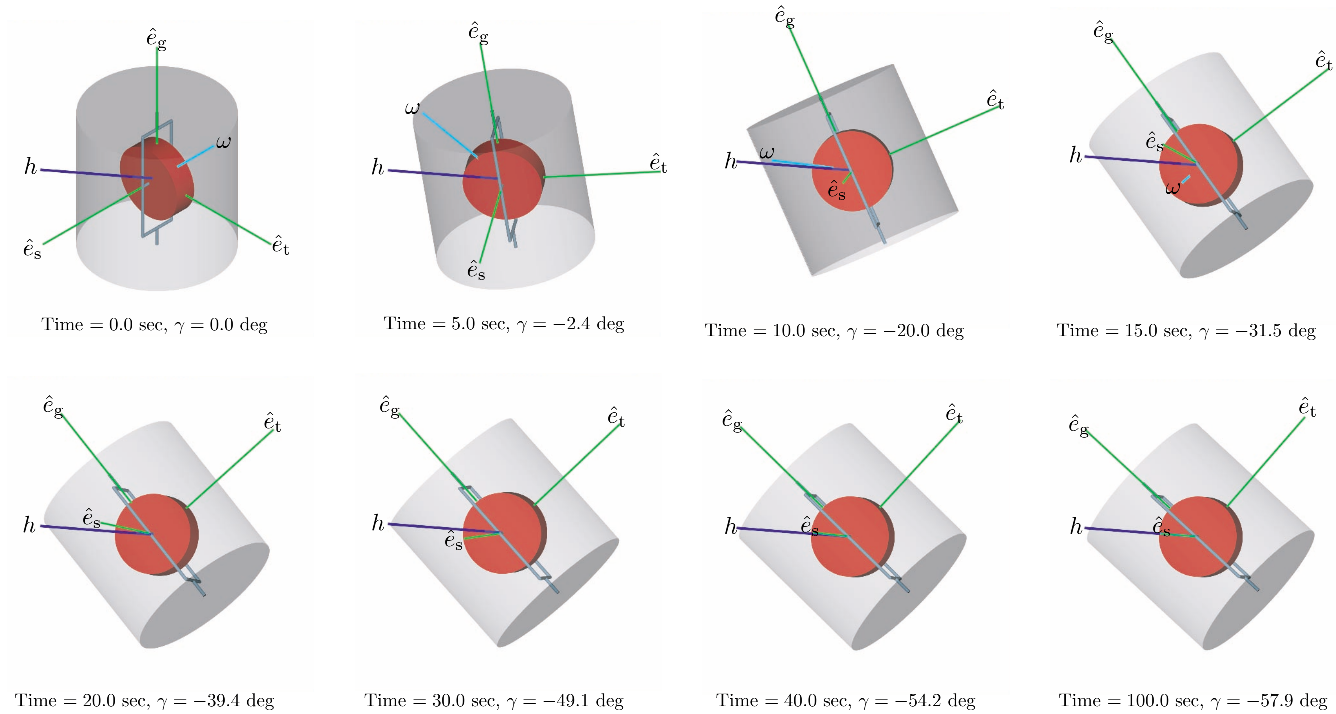

Fig. 9 Series of snapshots of a detumbling maneuver of a spacecraft with one VSMG. The gimbal frame unit vectors $\hat{e}_s$, $\hat{e}_t$, and $\hat{e}_g$, the angular momentum vector h, and the angular velocity vector ω are shown (not drawn to scale). The time of each snapshot and the gimbal angle γ are also depicted.

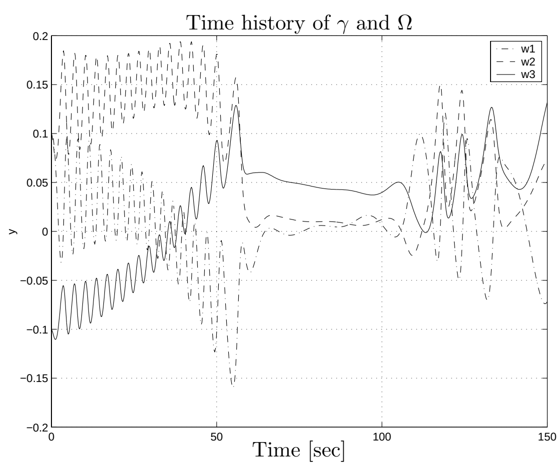

Fig. 8 Response of LQR controller with large initial conditions.

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